

# Tropical methods in representation theory

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We sketch some relations between tropical geometry and representation theory. This text originates from many discussions with Bernhard Keller and Alfredo Nájera, to whom I express my sincere gratitude.

Attention: this text serves as the memoir of third year at Ecole normale supérieure in Paris. As required, it is meant to be an introduction to a research area rather than an original research article.

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## 1 Canonical basis in Lie theory

### 1.1 Quiver representations

Representation theory of quivers is closely related to Lie theory. The first historical instance is

**Theorem 1.1** ([Gab72]). *A quiver has finitely many isomorphism classes of indecomposable representations if and only if it is an oriented Dynkin diagram. In this case there are bijections between the isomorphism classes of indecomposable representations, the dimension vectors of indecomposable representations and the set of positive roots of the Dynkin diagram.*

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More generally, we have

**Theorem 1.2** ([Kac80]). *For a finite quiver without vertex loops, the set of dimension vectors of the indecomposables over an algebraically closed field is in bijection with the set of positive roots of the underlying graph. Moreover, there is a unique indecomposable representation (up to isomorphism) corresponding to a given positive root if and only if this root is real.*

The second relation between quiver representations and Lie theory is through Ringel-Hall algebra.

**Definition 1.3.** The *Ringel-Hall algebra*  $\mathcal{R}$  for a quiver  $Q$  is the  $\mathbb{C}$ -vector space with basis elements  $V$  indexed by the isomorphism classes of representations of  $Q$ , and with structure constants defined in the following way. Given three representations  $V, V', V''$  of  $Q$ , the structural constant  $c_{V,V',V''}$  is the Euler characteristic in cohomology with compact support of the space of all short exact sequences of representations of the following type

$$0 \rightarrow V \rightarrow V'' \rightarrow V' \rightarrow 0.$$

**Theorem 1.4** ([Rin90]). *Let  $\Delta$  be a Dynkin diagram of type  $A, D, E$ ,  $\mathfrak{g}$  the associated complex semisimple Lie algebra,  $\mathfrak{n}$  a maximal nilpotent Lie subalgebra,  $U(\mathfrak{n})$  the enveloping algebra of  $\mathfrak{n}$ , and let  $e_1, \dots, e_n$  denote a basis of the root system in  $\mathfrak{n}$ . Let  $Q$  be a quiver whose underlying graph is  $\Delta$ , and let  $\mathcal{R}$  be the Ringel-Hall algebra for  $Q$ . Then there is a unique algebra isomorphism  $U(\mathfrak{n}) \simeq \mathcal{R}$  which takes each  $e_i \in U(\mathfrak{n})$  to the simple representation of  $Q$  corresponding to the vertex  $i$  of the Dynkin diagram  $\Delta$ .*

## 1.2 Canonical and semicanonical basis

We follow [Lus91] 19. Via the theorem above,  $U(\mathfrak{n})$  acquires from  $\mathcal{R}$  a  $\mathbb{C}$ -basis, which depends on the choice of orientation of the Dynkin diagram  $\Delta$ . Lusztig [Lus90] used perverse sheaves to remedy this dependence. He defined invariants  $j_{V,V'} \in \mathbb{N}$  for any two representations  $V, V'$  of the quiver  $Q$ . In the case where  $V, V'$  have different dimensions,  $j_{V,V'}$  is 0. Assume now that they have the same dimension vector  $\mathbf{d} = (d_1, \dots, d_n)$ . Let  $E_{\mathbf{d}}$  be the space of representations of dimension  $\mathbf{d}$  of  $Q$ . The group  $G_{\mathbf{d}} = \prod_i GL_{d_i}(\mathbb{C})$  acts on  $E_{\mathbf{d}}$  by conjugation. The two representations  $V, V'$  correspond to two  $G_{\mathbf{d}}$ -orbits  $\mathcal{O}, \mathcal{O}'$  in  $E_{\mathbf{d}}$ . If  $\mathcal{O}$  is contained in the closure of  $\mathcal{O}'$ ,  $j_{V,V'}$  is defined to be the Euler characteristic of the local intersection cohomology of that closure at a point in  $\mathcal{O}$ . Otherwise,  $j_{V,V'}$  is set to be 0.

**Theorem 1.5.** *The elements  $\{ \sum_V j_{V,V'} V \mid V' \text{ is a representation of } Q \}$  form a new basis of  $\mathcal{R}$ . Upon transferring it to  $U(\mathfrak{n})$ , we obtain a basis of  $U(\mathfrak{n})$  which is independent of the chosen orientation. This basis is called the canonical basis. Moreover, given an irreducible, finite dimensional representation  $V$  of  $\mathfrak{g}$  with a specified lowest weight vector. If we apply the elements of the canonical basis to this lowest weight vector, we obtain a canonical basis of the representation  $V$  with nice properties.*

Now we follow [Lec10] Section 4 and 5. In [Lus00], Lusztig introduced another basis  $\mathcal{S}$  of the enveloping algebra  $U(\mathfrak{n})$ , called the *semicanonical basis*. Using the duality between  $U(\mathfrak{n})$  and the coordinate ring  $\mathbb{C}[N]$  of  $N$ , one obtains the *dual semicanonical basis*  $\mathcal{S}^*$  of  $\mathbb{C}[N]$ .

Let us recall his construction briefly. Let  $\bar{Q}$  denote the quiver obtained from the Dynkin diagram  $\Delta$  by replacing every edge by a pair  $(\alpha, \alpha^*)$  of opposite arrows.

**Definition 1.6.** The *preprojective algebra*  $\Lambda$  is the quotient of the path algebra  $\mathbb{C}\bar{Q}$  by the two-sided ideal generated by the element

$$\rho = \sum (\alpha\alpha^* - \alpha^*\alpha),$$

where the sum is taken over all pairs  $(\alpha, \alpha^*)$  of opposite arrows.

Let  $S_i$  be the one-dimensional  $\Lambda$ -modules attached to the vertices  $i$  of  $\bar{Q}$ . For any sequence  $\mathbf{i} = (i_1, \dots, i_d)$ , any  $\Lambda$ -module  $X$  of dimension  $d$ , let  $\mathcal{F}_{X, \mathbf{i}}$  be the space of flags of submodules

$$0 = F_0 \subset F_1 \subset \dots \subset F_d = X$$

such that  $F_k/F_{k-1} \simeq S_{i_k}$  for  $k = 1, \dots, d$ . Denote by  $\Lambda_{\mathbf{d}}$  the space of  $\Lambda$ -modules  $X$  with a given dimension vector  $\mathbf{d} = (d_i)$  where  $\sum_i d_i = d$ . Let  $\chi_{\mathbf{i}}$  be the constructible function on  $\Lambda_{\mathbf{d}}$  given by  $\chi_{\mathbf{i}}(X) = \chi(\mathcal{F}_{X, \mathbf{i}})$  and let  $\mathcal{M}_{\mathbf{d}}$  be the  $\mathbb{C}$ -vector space spanned by the functions  $\chi_{\mathbf{i}}$  for all possible sequences  $\mathbf{i}$  of length  $d$ , and let  $\mathcal{M} = \bigoplus_{\mathbf{d} \in \mathbb{N}^n} \mathcal{M}_{\mathbf{d}}$ . Lusztig constructed an isomorphism  $\mathcal{M} \simeq U(\mathfrak{n})$ , and the semicanonical basis of  $U(\mathfrak{n})$  via this isomorphism. By duality between  $U(\mathfrak{n})$  and  $\mathbb{C}[N]$  as Hopf algebras, every object  $X \in \Lambda\text{-mod}$  gives rise to a polynomial function  $\varphi_X$  on  $N$ . Let  $\mathcal{I}_{\mathbf{d}}$  be the set of irreducible components of  $\Lambda_{\mathbf{d}}$  and  $\mathcal{I} = \coprod_{\mathbf{d}} \mathcal{I}_{\mathbf{d}}$ . Then the set  $\mathcal{S}^* = \{\varphi_Z \mid Z \in \mathcal{I}\}$  is the dual semicanonical basis, where  $\varphi_Z$  denotes the generic value of  $\varphi$  on  $Z$ .

## 2 Cluster structures

### 2.1 Cluster algebras and categorification

Cluster algebras were invented by Fomin and Zelevinsky [FZ02] in order to find a combinatorial algorithm for calculating the (semi)canonical basis constructed by Lusztig. Let us recall their definition and some related constructions in the case of skew-symmetric cluster algebras of geometric type following [Kel11].

**Definition 2.1.** Let  $1 \leq n \leq m$  be integers, and let  $\tilde{Q}$  be a quiver with  $m$  vertices numbered as  $1, \dots, m$  without loops or 2-cycles. We call the vertices  $n+1, \dots, m$  *frozen vertices* and the rest vertices *non-frozen vertices*.  $\tilde{Q}$  is called a *frozen quiver of type  $(n, m)$*  if there is no arrows between frozen vertices.

**Definition 2.2.** The cluster algebra  $\mathcal{A}_{\tilde{Q}}$  associated to a frozen quiver  $\tilde{Q}$  is a subalgebra of the field  $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_m)$  defined by a set of generators

constructed in the following iterative way. A *seed* is a pair  $(\tilde{R}, u)$  of a frozen quiver  $\tilde{R}$  of type  $(n, m)$  and a sequence of  $m$  variables  $u = (u_1, \dots, u_m)$  which generate freely the field  $\mathcal{F}$ . The first  $n$  variables  $u_1, \dots, u_n$  are called *cluster variables*, while  $u_{n+1}, \dots, u_m$  are called coefficients. The *mutation* of a seed  $(\tilde{R}, u)$  at a non-frozen vertex  $k$  of  $\tilde{R}$  gives a new seed  $(\tilde{R}', u')$ , where  $\tilde{R}'$  is obtained by quiver mutations (i.e. add a new arrow  $i \rightarrow j$  for every subquiver  $i \rightarrow k \rightarrow j$ , then reverse all arrows attached to  $k$ , then delete all 2-cycles), and  $u'$  is obtained from  $u$  by replacing the element  $u_k$  by the element  $u'_k$  defined by the exchange relation

$$u'_k u_k = \prod_{s(\alpha)=k} u_{t(\alpha)} + \prod_{t(\alpha)=k} u_{s(\alpha)},$$

where  $s(\alpha)$  and  $t(\alpha)$  denote respectively the source and the target of an arrow  $\alpha$  of the quiver  $\tilde{R}$ . We start with the initial seed  $(\tilde{Q}, (x_1, \dots, x_n))$ . A *cluster* associated to  $\tilde{Q}$  is a sequence  $u$  which appears in a seed  $(\tilde{R}, u)$  obtained from the initial seed by iterated mutations at non-frozen vertices. The *cluster algebra*  $\mathcal{A}_{\tilde{Q}}$  is the subalgebra of  $\mathcal{F}$  generated by all cluster variables. The *upper cluster algebra*  $\mathcal{U}_{\tilde{Q}}$  is the subalgebra of  $\mathcal{F}$  consisting of elements that can be expressed as Laurent polynomials in the variables in all clusters associated to  $\tilde{Q}$ . The full subquiver of  $\tilde{Q}$  consisting of all non-frozen vertices is called the *principal part* of  $\tilde{Q}$ , which we denote by  $Q$ .

**Theorem 2.3** ([FZ03]). *The number of clusters of  $\mathcal{A}_{\tilde{Q}}$  is finite if and only if the principal part  $Q$  is Dynkin. In this case, let  $\alpha_1, \dots, \alpha_n$  denote the simple roots. Then each positive root  $\alpha = d_1 \alpha_1 + \dots + d_n \alpha_n$  corresponds to a unique non-initial cluster variable  $X_\alpha$  of denominator  $x_1^{d_1} \dots x_n^{d_n}$ .*

Combining with Theorem 1.1, we obtain the one-to-one correspondence between indecomposable representations of  $Q$  and non-initial cluster variables. This gives a representation theoretic interpretation of the non-initial cluster variables. More generally, we have the following.

**Theorem 2.4** ([CC06]). *Let  $Q$  be an acyclic quiver. For any representation  $V$  of  $Q$ , define the Caldero-Chapoton map*

$$CC(V) = \frac{1}{x_1^{d_1} \dots x_n^{d_n}} \sum_e \chi(Gr_e(V)) \prod_{i=1}^n x_i^{\sum_{j \rightarrow i} e_j + \sum_{i \rightarrow j} (d_j - e_j)},$$

where  $(d_1, \dots, d_n)$  is the dimension vector of  $V$ , and  $Gr_e(V)$  denotes the Grassmannian of subrepresentations of  $V$  with dimension vector  $e$ . The map  $CC$  gives a bijection between rigid indecomposable representations of  $Q$  and non-initial cluster variables of  $\mathcal{A}_{\tilde{Q}}$ .

The Abelian category  $Q\text{-mod}$  can be enlarged to a 2-Calabi-Yau triangulate category  $\mathcal{C}_Q$  called the *cluster category*<sup>1</sup> ([BMR<sup>+</sup>06]). The Caldero-Chapoton map extended to  $\mathcal{C}_Q$  gives representation theoretic interpretations to all cluster variables, as well as exchange relations and the notion of clusters.

<sup>1</sup>For general quivers  $Q$  (not necessarily acyclic), there are methods of additive categorifications if we endow  $Q$  with a generic potential (see [Kel11] 4.4).

**Theorem 2.5** ([CK06]). *The Caldero-Chapoton map  $CC : \mathcal{C}_Q \rightarrow \mathcal{A}_Q$  has the following properties*

(i)  $CC(L \oplus M) = CC(L) \cdot CC(M)$  for any  $L, M \in \mathcal{C}_Q$ .

(ii) The map  $CC$  induces three bijections

$$\begin{aligned} \{\text{indecomposable rigid objects of } \mathcal{C}_Q\} &\xrightarrow{\sim} \{\text{cluster variables of } \mathcal{A}_Q\}, \\ \{\text{rigid objects of } \mathcal{C}_Q\} &\xrightarrow{\sim} \{\text{cluster monomials of } \mathcal{A}_Q\}, \\ \{\text{maximal rigid objects of } \mathcal{C}_Q\} &\xrightarrow{\sim} \{\text{clusters of } \mathcal{A}_Q\} \end{aligned}$$

(iii) Two indecomposable rigid objects  $L, M$  correspond to an exchange pair if and only if  $\dim \text{Ext}^1(L, M) = 1$ . If this is the case, and let

$$L \rightarrow E \rightarrow M \xrightarrow{+1}, \quad M \rightarrow E' \rightarrow L \xrightarrow{+1}$$

be two non-split triangles. We have the exchange relation

$$CC(L) \cdot CC(M) = CC(E) + CC(E').$$

(iv) Let  $T$  be a maximal rigid object of  $\mathcal{C}_Q$  corresponding to a cluster  $(R, (u_1, \dots, u_n))$ . Then the quiver associated<sup>2</sup> to the endomorphism algebra  $\text{End}_{\mathcal{C}_Q}(T)$  is the quiver  $R$ .

## 2.2 Cluster monomials and canonical basis

Now we use notations in Section 1.2. The coordinate ring  $\mathbb{C}[N]$  can be endowed with the structure of an upper cluster algebra with coefficients ([BFZ05]). Then our map  $\varphi : \Lambda\text{-mod} \rightarrow \mathbb{C}[N]$  is analogous to the Caldero-Chapoton map. Let  $T = T_1 \oplus \dots \oplus T_m$  be a maximal rigid  $\Lambda$ -module, where every  $T_i$  is indecomposable. Denote by  $\Gamma_T$  the quiver associated to the endomorphism algebra  $\text{End}_\Lambda T$ . Define  $\Sigma(T) = (\Gamma_T, (\varphi_{T_1}, \dots, \varphi_{T_m}))$ .

**Theorem 2.6** ([GLS07]). *There exists an explicit maximal right  $\Lambda$ -module  $U$  such that  $\Sigma(U)$  is one of the seeds of the cluster structure on  $\mathbb{C}[N]$ .*

The notion of seed mutation can be lifted to the category  $\Lambda\text{-mod}$ .

**Theorem 2.7** ([GLS06]). *Let  $T_k$  be a non-projective indecomposable summand of  $T$ . There exists a unique indecomposable module  $T_k^* \not\cong T_k$  such that  $(T/T_k) \oplus T_k^*$  is maximal rigid.*

A rigid  $\Lambda$ -module is called *accessible* if its indecomposable direct summands are obtained from iterated mutations from the initial seed constructed in Theorem 2.6.

<sup>2</sup>By definition, the quiver associated to an algebra has vertices corresponding to simple modules over this algebra, and the numbers of arrows between two vertices is the dimension of the group  $\text{Ext}^1$  between the corresponding modules.

**Theorem 2.8** ([GLS06]). *We have the analogue of Theorem 2.5.*

(i)  $\varphi_{L \oplus M} = \varphi_L \varphi_M$ .

(ii) *The map  $\varphi$  induces three bijections*

$$\begin{aligned} \{\text{accessible indecomposable rigid objects of } \Lambda\text{-mod}\} &\xrightarrow{\sim} \{\text{cluster variables of } \mathcal{A}_Q\}, \\ \{\text{accessible rigid objects of } \Lambda\text{-mod}\} &\xrightarrow{\sim} \{\text{cluster monomials of } \mathcal{A}_Q\}, \\ \{\text{accessible maximal rigid objects of } \Lambda\text{-mod}\} &\xrightarrow{\sim} \{\text{clusters of } \mathcal{A}_Q\} \end{aligned}$$

(iii) *Two indecomposable rigid objects  $L, M$  correspond to an exchange pair if and only if  $\dim \text{Ext}^1(L, M) = 1$ . If this is the case, and let*

$$0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0, \quad 0 \rightarrow M \rightarrow E' \rightarrow L \rightarrow 0$$

*be two non-split exact sequences, we have the exchange relation*

$$\varphi_L \varphi_M = \varphi_E + \varphi_{E'}.$$

(iv) *We have  $\Sigma(\mu_k(T)) = \mu_k(\Sigma(T))$ , where  $\mu_k$  denotes the mutation at place  $k$ .*

**Corollary 2.9.** *The cluster monomials in  $\mathbb{C}[N]$  is part of the dual semi-canonical basis.*

*Remark 2.10.* One can ask whether cluster monomials belong to the canonical basis.

*Remark 2.11.* One can study the quantum analog of the previous statements.

### 3 Heuristics from symplectic geometry

Cluster structures is a particular case of the more general notion of wall-crossing structures ([KS13]). Let us explain some heuristics from symplectic geometry following [GS12] and [GHK11]. Let  $X$  and  $\check{X}$  be a pair of mirror Calabi-Yau  $n$ -folds. Their geometry is suggested by the following two insightful conjectures.

**Strominger-Yau-Zaslow (SYZ) conjecture** There exists special Lagrangian fibrations  $\tau : X \rightarrow B$ ,  $\check{\tau} : \check{X} \rightarrow B$  over real  $n$ -dimensional manifold  $B$ , whose generic fibers are tori. The fibration  $\tau$  induces two dual  $\mathbb{Z}$ -affine structures  $\nabla^J$ ,  $\nabla^\omega$  on an open dense part  $B_0 \subset B$ , the former holomorphic in nature, the latter symplectic in nature. Let  $\check{\nabla}^J$  and  $\check{\nabla}^\omega$  denote the dual  $\mathbb{Z}$ -affine structures induced by the fibration  $\check{\tau}$ . We have  $\nabla^J \simeq \check{\nabla}^\omega$  and  $\nabla^\omega \simeq \check{\nabla}^J$ .

**Kontsevich’s homological mirror symmetry (HMS) conjecture** The Fukaya category  $Fuk(\check{X})$  of  $\check{X}$  is equivalent to the bounded derived category  $D^b(X)$  of coherent sheaves on  $X$  with  $A_\infty$ -enhancement.

We cannot explain precisely the various notions mentioned in the conjectures because part of the game is to find the right definition for the conjectures to hold. Nevertheless, the two conjectures combined together, gives birth to the notion of wall-crossing structures, which sheds much light on the theory of cluster algebras and canonical basis, as well as the theory of integrable systems.

Under homological mirror symmetry, special Lagrangian torus fibers of  $\check{\tau}$  correspond to skyscraper sheaves on  $X$ , while Lagrangian sections of  $\check{\tau}$  corresponds to line bundles on  $X$ . Suppose  $L_0$  is a Lagrangian section of  $\check{\tau}$  corresponding to the structure sheaf  $\mathcal{O}_X$ , and  $L_1$  is a Lagrangian section corresponding to an ample line bundle  $\mathcal{L}$  on  $X$ . Then HMS should yield an isomorphism  $\text{Hom}(L_0, L_1) \simeq \text{Hom}(\mathcal{O}_X, \mathcal{L})$ . In nice situations when all higher cohomology  $H^*(X, \mathcal{L})$  vanishes and all the intersection points between  $L_0$  and  $L_1$  are Maslov index zero, the obvious basis for  $\text{Hom}(L_0, L_1)$  given by the intersection points induces a “canonical” basis for  $H^0(X, \mathcal{L})$ , whose elements are usually called *theta functions*.

Certainly the “canonical basis” depends on the choice of Lagrangian sections. There is in fact a whole set of natural sections  $L_l$ , indexed by  $l \in \mathbb{Z}$ , given in local integral affine coordinates on  $B$  by

$$\sigma_l : (y_1, \dots, y_n) \mapsto - \sum_{i=1}^n l \cdot y_i \frac{\partial}{\partial y_i}.$$

The set of intersection points  $L_0 \cap L_l$  is then parametrized by the lattice points  $B_0(\frac{1}{l}\mathbb{Z})$ . Let  $m \in B_0(\frac{1}{l}\mathbb{Z})$ , so  $m$  determines a theta function  $\theta_m \in \text{Hom}(\mathcal{O}_X, \mathcal{L}^{\otimes l})$ . We would like to specify the value of  $\theta_m$  at a point  $x \in X$ . A non-zero element of  $\text{Hom}(\mathcal{L}^{\otimes l}, \mathcal{O}_x)$  gives an identification  $\mathcal{L}^{\otimes l} \otimes \mathcal{O}_x \simeq \mathcal{O}_x$ . Then the composition of  $\theta_m$  with this identification under

$$\text{Hom}(\mathcal{L}^{\otimes l}, \mathcal{O}_x) \otimes \text{Hom}(\mathcal{O}_X, \mathcal{L}^{\otimes l}) \rightarrow \text{Hom}(\mathcal{O}_X, \mathcal{O}_x) \quad (3.1)$$

gives the value of  $\theta_m$  at  $x$ . Under HMS, (3.1) corresponds to the Floer multiplication

$$\mu_2 : \text{Hom}(L_l, L_x) \otimes \text{Hom}(L_0, L_l) \rightarrow \text{Hom}(L_0, L_x),$$

where  $L_x$  denotes the fibre of  $\check{\tau}$  over  $x$ . This suggests that the theta function  $\theta_m$  should be defined as a sum over Maslov index 2 disks with boundary on the Lagrangians  $L_l, L_x, L_0$ . Moreover, the exponentials on the summands should correspond to the symplectic area of the holomorphic disks.

For the theta function  $\theta_m$  to make sense, the moduli space of Maslov index 2 disks considered above must deform smoothly with the Lagrangian  $L_x$ . This fails for Lagrangians that bound Maslov index 0 disks. Such Lagrangians are said to be obstructed. The locus on  $B$  corresponding to obstructed Lagrangians is by definition the union of the walls, which is of real codimension 1. When we cross the wall the theta function  $\theta_m$  are discontinuous. The discontinuity is corrected

by a holomorphic change of variables in the local coordinates according to the data associated to the walls determined by the counting of Maslov index 0 disks.

Nevertheless, there are huge technical difficulties regarding holomorphic disk counting. In general, the number of the disks mentioned above is not rigorously defined, let alone study their properties. It is suggested in [KS01] that the special Lagrangian fibration resembles a lot the deformation retraction of a  $k$ -analytic space to its skeleton constructed by Vladimir Berkovich ([Ber99]). Therefore, techniques from non-Archimedean geometry can be used to study the counting of disks. This is an ongoing project between myself and my advisor Kontsevich. [Yu13a][Yu13b] are very preliminary results in this direction.

On the other hand, it is proposed in [KS06] that one can circumvent the counting of Maslov index zero disks using the a posteriori wall-crossing formula. As for Maslov index two disks, Mark Gross et al. ([GHKS]) invented the notion of jagged paths, which is the tropical analogue of such disks, thus can be counted combinatorially.

In our description in the symplectic setting, we must assume  $l \neq 0$ . Now for the combinatorially defined notion of jagged path, it makes sense to specialize to the case  $l = 0$ , and one arrives at the notion of *broken lines* (see Definition 4.4). In a word, jagged paths control the propagation of local monomial sections of  $\mathcal{L}^{\otimes l}$ , while broken lines control the propagation of local monomials on  $X$  itself.

One can also provide a geometric interpretation of broken lines. A broken line with initial direction  $q$  and endpoint  $x$  can be thought of as the tropicalization of a Maslov index 2 disk which intersects one time with a specified boundary divisor  $E_q$  in a certain minimal model of  $\check{X}$ , and whose boundary lie in the fiber  $L_x$  and passes through a given point in  $L_x$ . Counting of such disks gives the obstruction co-chain of the Lagrangian  $L_x$ . They should correspond to Landau-Ginzburg superpotentials on the mirror (see for example [Tom01]), which are in particular holomorphic functions.

There is yet another way to construct holomorphic functions on  $X$  rather than sections of  $\mathcal{L}^{\otimes l}$  for  $l \neq 0$  proposed by Kontsevich during his lectures at Jussieu in April 2012. One can use heuristics from wrapped Fukaya category. Let  $L_t$  denote the perturbation of  $L_0$  under the Hamiltonian flow associated to a certain quadratic potential  $\phi$ . We obtain

$$\mathrm{Hom}(L_0, L_0) \simeq \mathrm{Hom}(L_0, L_t), \quad \text{for } t > 0.$$

One imagines the intersection points between  $L_0$  and  $L_t$  as the lattice points  $B_0(\mathbb{Z})$ . Under the assumption that  $\partial^2 \phi|_{B_0} > 0$ , the Floer complex above is concentrated in degree zero. So we can proceed as in the case of line bundles. That is, we obtain the composition map

$$\mu_2 : \mathrm{Hom}(L_0, L_0) \otimes \mathrm{Hom}(L_0, L_x) \rightarrow \mathrm{Hom}(L_0, L_x)$$

which corresponds under HMS to the composition map

$$\mathrm{Hom}(\mathcal{O}, \mathcal{O}) \otimes \mathrm{Hom}(\mathcal{O}, \mathcal{O}_x) \rightarrow \mathrm{Hom}(\mathcal{O}, \mathcal{O}_x).$$

The tropicalizations of the disks contributing to the composition map  $\mu_2$  above also have combinatorial descriptions. They are explained as paths of lights in [Yu12] 4.6. Counting of such gadgets produces for us holomorphic functions on the mirror. One can ask whether the canonical theta functions produced by broken lines and those produced by light paths are the same. Denis Auroux explained to me that one can expect an isomorphism (or an injection) from the wrapped Floer homology groups to symplectic cohomology ([Abo10])

$$HW_*(L_0, L_0) \rightarrow SH^*(\check{X}).$$

Then regard  $\text{Hom}(L_0, L_x)$  as a module over  $HW_*(L_0, L_0)$ . There is also some construction that makes  $H_*(L_x)$  (to confirm later) a module over  $SH^*(\check{X})$ . So the coincidence between the results produced by broken lines and light paths should be explained by the isomorphism between the two modules. It may be interesting to make the arguments above more clear.

## 4 Counting tropical curves

Heuristics in the last section lead to interesting applications of tropical geometry to cluster algebras. Let us review some of the ideas based on the lectures given by Sean Keel in Strasbourg in June 2013.

We start with two observations.

*Observation.* A seed for a cluster algebra of geometric type can be encoded in the following datum: a lattice  $N$  of rank  $n$ , a skew-symmetric form  $\langle, \rangle : \Lambda^2 N \rightarrow \mathbb{Q}$ , a basis  $(e_1, \dots, e_n)$  of  $\mathbb{N}$ , and possibly a declaration that some of the base vectors are frozen.

Let us denote  $M = N^*$ .

*Observation.* The vector space  $M_{\mathbb{R}} = M \otimes \mathbb{R}$  can be regarded as the tropicalization of the cluster  $\mathcal{X}$ -variety. Mutations are birational transformations on the cluster  $\mathcal{X}$ -variety. They give rise to piecewise linear transformations on  $M_{\mathbb{R}}$ . (Very detailed explanations are given in [Yu12] Section 4.4).

Now the idea is to do tropical geometry on  $M_{\mathbb{R}}$  to study the cluster algebra  $\mathcal{A}$ . We would like to have a canonical basis of  $\mathcal{A}$  parametrized by the lattice  $M$ . Let us review some constructions in tropical geometry. In order for the tropical techniques to work, one need the assumption that  $v_i = \langle e_i, - \rangle, i = 1, \dots, n$  span a strictly convex cone  $C \subset M_{\mathbb{R}}$ . This should also be verified under all possible mutations. Otherwise, there is the problem of taking the completion  $\widehat{\mathbb{Q}[C]}$  of the ring  $\mathbb{Q}[C]$  with respect to the maximal ideal generated by monomials. One can verify that the assumption is fulfilled in the case of cluster algebra with principal coefficients.

**Definition 4.1.** A wall  $W$  is a full dimensional convex cone  $W \subset e_W^\perp$  for some primitive vector  $e_W \in N$ , equipped with a monomial  $m_W \in e_W^\perp \cap C \subset M_{\mathbb{R}}$ , and a function  $f_W = 1 + \sum_{r \geq 1} c_r \cdot z^{r \cdot m_W} \in \widehat{\mathbb{Q}[C]}$ . If  $W + \mathbb{R}_{\geq 0} m_W \subset W$ , it is called an *incoming wall*, otherwise it is called an *outgoing wall*. A *scattering diagram*  $\mathcal{D}$  is a collection of walls with certain adic convergence assumptions.

Crossing a wall  $W$  gives an automorphism  $T_W$  of  $\widehat{\mathbb{Q}[C]}$  defined as

$$z^v \mapsto z^v \cdot (f_W)^{[v, e_W]}.$$

When we have a path  $\Gamma$  transversal to all walls, we can take composition  $\mu_\Gamma$  of all such automorphisms.

**Definition 4.2.** A scattering diagram  $\mathcal{D}$  is called *consistent* if  $\mu_\Gamma$  depends only on the starting point and the endpoint.

**Theorem 4.3** ([KS06][GS11]). *Given any configuration of incoming walls, one can add possibly infinitely many outgoing walls in a unique way to obtain a consistent scattering diagram.*

Let  $\{ (e_i^\perp, 1 + z^{v_i}) \mid i = 1, \dots, n \}$  be the incoming walls. The theorem above produces for us a scattering diagram  $\mathcal{D}$ . It is easy to see that this procedure is compatible with mutations.

Now let us define the tropical incarnation of Maslov index 2 disks.

**Definition 4.4.** A *broken line* is a piecewise linear map from  $(-\infty, 0]$  to  $M_{\mathbb{R}}$  such that

- (i) There are finitely many linear segments. We order them by  $1, 2, \dots, n_\gamma$ .
- (ii) Each linear segment  $k$  is equipped with a monomial  $c_k \cdot z^{q_k}$ , for some  $q_k \in M$ . We denote  $c(\gamma) = c_{n_\gamma}$ ,  $F(\gamma) = z^{q_{n_\gamma}}$ .
- (iii) We require that  $-q_k$  is parallel to the edge  $k$  for  $k = 1, \dots, n_\gamma$ .
- (iv) We require that  $c_1 = 1$ , and we call  $q_1$  the *initial direction* of  $\gamma$ .
- (v) Bends of  $\gamma$  can only occur at walls. Suppose that the segment  $k$  and the segment  $k + 1$  are separated by a wall  $W$ . Then there exists a monomial term  $L$  in the expansion of  $f_W^{[e, q_k]}$  such that  $c_{k+1} z^{q_{k+1}} = L \cdot c_k z^{q_k}$ .

Finally, we can explain how broken lines give rise to canonical basis. Pick a point  $b$  in a chamber of the scattering diagram  $\mathcal{D}$  free from walls. Then for any  $q \in M$ , we obtain a canonical theta function

$$\theta_{b,q} = \sum_{\gamma} c(\gamma) z^{F(\gamma)},$$

where the sum is taken over all broken lines  $\gamma$  with initial direction  $q$ , and which terminates at the point  $b$ . If we move the point  $b$  in the same chamber,  $\theta_{b,q}$  remains unchanged. Otherwise, it changes according to mutation rules.

Using broken lines, one can also obtain the structural constants of the multiplication rule of the canonical theta functions. Again it has a geometric interpretation as counting rational curves in a certain minimal model of the cluster  $\mathcal{X}$ -variety with given intersection multiplicities with boundary divisors. But I don't know why counting such rational curves gives us multiplication rules.

*Remark 4.5.* One can ask the relation between canonical theta functions constructed here and the (semi)canonical basis constructed by Lusztig.

*Remark 4.6.* Sean Keel asked for a representation theoretic interpretation of broken lines. More generally, one can try to build a dictionary between objects in tropical geometry and objects in representation theory. The link may be established via Donaldson-Thomas invariants.

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